MATH4210: Financial Mathematics Tutorial 9

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Question (a)

We consider a continuous time market, where the interest rate $r = 0$, and the risky asset $S = (S_t)_{0 \le t \le T}$ follows the Balck-Scholes model with initial value $S_0 = 1$, drift μ and volatility $\sigma > 0$ (without any dividend), so that

$$S_t = S_0 \exp\,(\mu - \sigma^2/2)t + \sigma B_t$$

$dS_t = \mu S_t dt + \sigma S_t dB_t$

Solve the following questions:

(a) A self-financing portfolio is given by (x, ϕ), where x represents the initial wealth of the portfolio, and ϕ_t represents the number of risky asset in the portfolio at time t. Let $\Pi_t^{x,\phi}$ be the wealth process of the portfolio, write down the dynamic of $\Pi^{x,\phi}$ in $t \in [0, T]$ in form of

$$d\Pi_t^{x,\phi} = \alpha_t dt + \beta_t dB_t.$$

$d\Pi_t = \cancel{(\Pi_t - \phi_t S_t) r dt} + \phi_t dS_t$

$= \phi_t(\mu S_t dt + \sigma S_t dB_t)$

$= \mu \phi_t S_t dt + \sigma \phi_t S_t dB_t$

Find α and β.

Continuous Market Models

Question (b)

(b) There exists a unique risky-neutral probability $\mathbb{Q}$, together with a Brownian motion $B^{\mathbb{Q}}$ under the probability measure $\mathbb{Q}$. Give the expression of S_t as a function of $(t, B_t^{\mathbb{Q}})$.

$\cancel{\mu}$

under $\mathbb{Q}$: $dS_t = r \cdot S_t dt + \sigma S_t dB_t^{\mathbb{Q}}$

$$\Rightarrow \quad S_t = S_0 \exp\left(\left(r - \frac{\sigma^2}{2} \right) t + \sigma B_t^{\mathbb{Q}} \right)$$

0

$$dS_t = \mu S_t dt + \sigma S_t dB_t$$

Question (c)

(c) We first consider a derivative option with payoff $g(S_T) = S_T^2$ *at maturity* T.

(i) Compute the value

$$V_0 = \mathbb{E}^{\mathbb{Q}}[S_T^2].$$

$$V_0 = E^Q[S_T^2]$$

$$= E^Q[S_0^2 \exp 2((r - \tfrac{\sigma^2}{2})T + \sigma B_T)] \quad r = 0$$

$$= E^Q[\exp(-\tfrac{\sigma^2}{2} \cdot 2T + 2\sigma B_T)]$$

$$= e^{-\sigma^2 T} \, E^Q[e^{2\sigma B_T}] \qquad 2\sigma B_T \sim N(0, 4\sigma^2 T)$$

$$\left(X \sim N(\mu, \sigma^2), \quad E(e^X) = e^{\mu + \frac{\sigma^2}{2}} \right)$$

$$= e^{-\sigma^2 T} \cdot e^{0 + \frac{4\sigma^2 T}{2}} = e^{\sigma^2 T}$$

Continuous Market Models

$\partial_t v = x^2 \exp \sigma^2(T-t) \cdot (-\sigma^2)$

$\partial_x v = 2x \cdot \exp \sigma^2(T-t)$

$\partial_{xx} v = 2 \cdot \exp \sigma^2(T-t)$

Question (c)

(ii) Let $v(t,x) := x^2 \exp \sigma^2(T-t)$, *compute* $\partial_t v$,$\partial_x v$ *and* $\partial^2_{xx} v$. *Check that* v *satisfies the equation*

$x^2 \exp \sigma^2(T-T) = x^2$

$$\partial_t v(t,x) + \frac{1}{2}\sigma^2 x^2 \partial^2_{xx} v(t,x) = 0, \quad v(T,x) = x^2.$$

$x^2 \cdot (-\sigma^2) \exp \sigma^2(T-t) + \cancel{\frac{1}{2}} \sigma^2 x^2 \cdot \cancel{2} \cdot \exp \sigma^2(T-t) = 0$

Continuous Market Models

$dt \sim o(t)$ $dB_t \sim O(\sqrt{dt})$ $O(t^{\frac{1}{2}})$

$dv(t, S_t) = \partial_t v\,dt + \partial_x v\,dS_t + \frac{\partial_{xx} v}{2}(dS_t)^2$

$= \cancel{\partial_t v\,dt} + \partial_x v(\mu S_t dt + \sigma S_t dB_t) \cancel{+ \frac{\partial_{xx} v}{2}\sigma^2 S_t^2 dt} = \partial_x v \cdot dS_t$

$dS_t = \mu S_t dt + \sigma S_t dB_t$

$\Rightarrow (dS_t)^2 = \cancel{\mu^2 S_t^2 (dt)^2} + \sigma^2 S_t^2 (dB_t)^2 \cancel{+ \mu\sigma S_t^2 dB_t dt}$ 2 dt 1.5

Question (c)

(iii) Remember that S_t is a function of (t, B_t), apply the Ito formula on $v(t, S_t)$ to deduce that

$v(t,x) = x^2 \exp \sigma^2(T-t)$

$$S_T^2 = V_0 + \int_0^T \phi_t dS_t, \quad \text{where} \quad \phi_t := \partial_x v(t, x).$$

Then deduce that V_0 is the (no-arbitrage) price of the derivative option $g(S_T) = S_T^2$.

$v(t,x)$ satisfies BS ; $v(T,x) = g(x)$

$\int_0^T \Rightarrow v(T, S_T) - v(0, 1) = \int_0^T \partial_x v\,dS_t$

$\Rightarrow v(t, S_t)$ wealth process perfect replication of an option with payoff $g(S_T)$

$v(T,S_T) = S_T^2$, $v(0,1) = \exp \sigma^2 T = V_0$

Question (d)

(d) We now consider another option with (path-dependent) payoff $\int_0^T S_t^2 dt$.

(i) Remember that S_t *is a function of* (t, B_t), *apply the Ito formula to deduce that*

$$S_T^2 = S_0^2 + \int_0^T 2S_t dS_t + \sigma^2 \int_0^T S_t^2 dt.$$

$$dS_t^2 = 2S_t dS_t + \frac{2}{2}\underbrace{(dS_t)^2}_{\sigma^2 S_t^2 dt}$$

$$= 2S_t dS_t + \sigma^2 S_t^2 dt$$

$$\xRightarrow{\int_0^T} S_T^2 - S_0^2 = \int_0^T 2S_t dS_t + \sigma^2 \int_0^T S_t^2 dt$$

Question (d)

(ii) From the above, one obtains that

$$\sigma^2 \int_0^T S_t^2 dt = S_T^2 - S_0^2 - \int_0^T 2S_t dS_t.$$

Deduce the replication cost and replication strategy of the derivative option $\int_0^T S_t^2 dt$. (Hint: Use the above replication strategy for the option $g(S_T) = S_T^2$.)

$-v(T-t)$

$v(t,x) = E^Q\left[\int_0^T S_t^2 dt \mid S_t = x\right]$

$\Rightarrow v(T, S_T) = v(0, S_0) + \int_0^T \phi_t dS_t$

$\int_0^T S_t^2 dt$

$v_0 + \int_0^T \partial_x v \, dS_t$

$$\int_0^T S_t^2\,dt = \frac{S_T^2 - S_0^2}{\sigma^2} - \frac{1}{\sigma^2}\int_0^T 2S_t\,dS_t$$

$$= \frac{\overset{S_0^2 \exp \sigma^2 T}{V_0} - S_0^2}{\sigma^2} + \frac{1}{\sigma^2}\int_0^T \partial_x v\,dS_t - \frac{1}{\sigma^2}\int_0^T 2S_t\,dS_t$$

$$\frac{1}{\sigma^2}\int_0^T (\partial_x v - 2S_t)\,dS_t$$

check:

$$v(0, \overset{=1}{S_0}) = E^Q\left[\int_0^T S_t^2\,dt \,\middle|\, S_0 = 1\right]$$

$$= E^Q\left[\int_0^T S_0^2 \exp\left(2\left(\left(\overset{=0}{r} - \frac{\sigma^2}{2}\right)t + \sigma B_t\right)\right) dt \,\middle|\, S_0 = 1\right]$$

$$= E^Q\left[\int_0^T \exp(-\sigma^2 t + 2\sigma B_t)\,dt\right]$$

$$= \int_0^T e^{-\sigma^2 t} \cdot \underbrace{E^Q\left[e^{2\sigma B_t}\right]}_{e^{2\sigma^2 t}} dt \qquad 2\sigma B_t \sim N(0, 4\sigma^2 \cdot t)$$

$$= \int_0^T e^{\sigma^2 t}\,dt$$

$$= \frac{e^{\sigma^2 t}}{\sigma^2}\Big|_0^T = \frac{e^{\sigma^2 T} - 1}{\sigma^2}$$