MATH4210: Financial Mathematics Tutorial 9

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Question (a)

We consider a continuous time market, where the interest rate r=0, and the risky asset $S=(S_t)_{0\leq t\leq T}$ follows the Balck-Scholes model with initial value $S_0=1$, drift μ and volatility $\sigma>0$ (without any dividend), so that

$$S_t = S_0 \exp(\mu - \sigma^2/2)t + \sigma B_t e^{dS_t} = \mu S_t dt + 6 S_t dB_t$$

Solve the following questions:

(a) A self-financing portfolio is given by (x, ϕ) , where x represents the initial wealth of the portfolio, and ϕ_t represents the number of risky asset in the portfolio at time t. Let $\Pi_t^{x,\phi}$ be the wealth process of the portfolio, write down the dynamic of $\Pi^{x,\phi}$ in $t \in [0,T]$ in form of

$$d\Pi_{t}^{x,\phi} = \alpha_{t}dt + \beta_{t}dB_{t}.$$

$$= \phi_{t}(uS_{t}dt + 6S_{t}dB_{t})$$

$$= \phi_{t}(uS_{t}dt + 6S_{t}dB_{t})$$



Question (b)

(b) There exists a unique risky-neutral probability \mathbb{Q} , together with a Brownian motion $B^{\mathbb{Q}}$ under the probability measure \mathbb{Q} . Give the expression of S_t as a function of $(t, B_t^{\mathbb{Q}})$.

under
$$Q : dSt = Y \cdot Stdt + 6StdBt$$

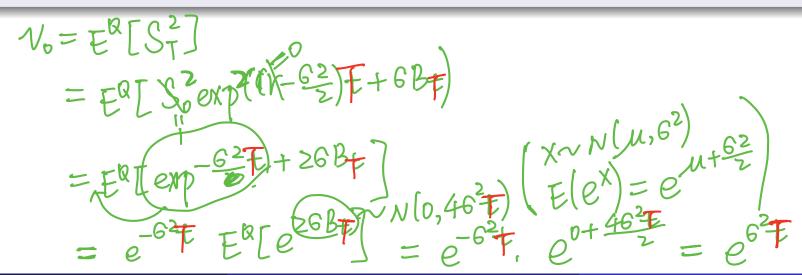
$$\Rightarrow St = S_0 exp(Y - S_2^2)t + 6Bt$$

$$dSt = uStdt + 6StdBt$$

Question (c)

- (c) We first consider a derivative option with payoff $g(S_T) = S_T^2$ at maturity T.
- (i) Compute the value

$$V_0 = \mathbb{E}^{\mathbb{Q}}[S_T^2].$$



$$\partial_{t}v = \chi^{2} \exp^{G^{2}(T-t)}, (-6^{2})$$

$$\partial_{x}v = 2\chi \cdot \exp^{G^{2}(T-t)}$$

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Question (c)

(ii) Let $v(t,x) := x^2 \exp \sigma^2(T-t)$, compute $\partial_t v_i \partial_x v_i$ and $\partial_{xx}^2 v_i$. Check that $X^{2} \exp^{6^{2}(T-T)} = X^{2}$ v satisfies the equation

$$\frac{\partial_t v(t,x) + \frac{1}{2}\sigma^2 x^2 \partial_{xx}^2 v(t,x) = 0}{\chi^2 (-6^2) \exp^{6^2(T-t)} + \frac{1}{2}6^2 \chi^2, \ \chi \cdot \exp^{6^2(T-t)} = 0}$$



$$dv(t,S_t) = \partial_t V dt + \partial_x V dS_t + \frac{\partial_x x V}{2} (dS_t)^2$$

$$= \partial_t V dt + \partial_x V (uS_t dt + 6S_t dS_t)$$

$$+ \partial_x V (uS_t dt + 2xV dS_t)$$

$$dS_{t} = \mu S_{t}dt + 6S_{t}dB_{t}$$

$$= |dS_{t}|^{2} = \mu^{2}S_{t}^{2}(dt)^{2} + 6^{2}S_{t}^{2}(dB_{t})^{2} + 46S_{t}^{2}(dB_{t})^{2} + 46S_{t}^{2}(dB_{t})^{2}$$

Question (c)

(iii) Remember that S_t is a function of (t, B_t) , apply the Ito formula on $v(t, S_t)$ to deduce that $V(t, S_t) = V(t, S_t)$

$$S_T^2 = V_0 + \int_0^T \phi_t dS_t$$
, where $\phi_t := \partial_x v(t, x)$.

Then deduce that V_0 is the (no-arbitrage) price of the derivative option

$$g(S_T)=S_T^2.$$

$$V(T, S_T) - V(0, 1) = \int_0^T \partial_x V dS_t$$

$$S_T^2 = \exp^{6^2 T}$$

Question (d)

- (d) We now consider another option with (path-dependent) payoff $\int_0^T S_t^2 dt$.
- (i) Remember that S_t is a function of (t, B_t) , apply the Ito formula to deduce that

$$S_T^2 = S_0^2 + \int_0^T 2S_t dS_t + \sigma^2 \int_0^T S_t^2 dt.$$

$$dS_{t}^{2} = 2S_{t}dS_{t} + \frac{2}{2}[dS_{t}]^{2} 6^{2}S_{t}^{2}dt$$

$$= 2S_{t}dS_{t} + 6^{2}S_{t}^{2}dt$$

$$S_{T}^{T} - S_{o}^{2} = \int_{0}^{T} z S_{t} dS_{t} + 6^{2} \int_{0}^{T} S_{t}^{2} dt$$

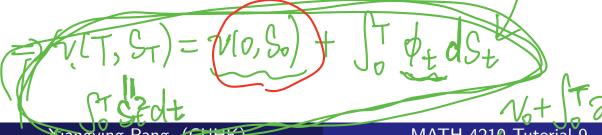
Question (d)

(ii) From the above, one obtains that

$$\sigma^2 \int_0^T S_t^2 dt = S_T^2 - S_0^2 - \int_0^T 2S_t dS_t.$$

Deduce the replication cost and replication strategy of the derivative option $\int_0^1 S_t^2 dt$. (Hint: Use the above replication strategy for the option

$$g(S_T) = S_T^2.)$$



$$\int_{0}^{T} S_{t}^{2} dt = \frac{S_{1}^{2} - S_{0}^{2}}{S_{0}^{2}} - \frac{1}{6^{2}} \int_{0}^{T} 2S_{t} dS_{t}$$

$$= \frac{V_{0} - S_{0}^{2}}{S_{0}^{2}} + \frac{1}{6^{2}} \int_{0}^{T} 2S_{t} dS_{t}$$

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$$= \frac{V_{0} - S_{0}^{2}}{S_{0}^{2}} + \frac{1}$$